Growth Estimates on Positive Solutions of the Equation $\Delta u + K u^{\frac{n+2}{n-2}} = 0$ in $\mathbb{R}^n$

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Abstract

We construct unbounded positive C^2-solutions of the equation $\Delta u + Ku^{(n+2)/(n-2)} = 0$ in $\mathbb{R}^n$ (equipped with Euclidean metric g_o) such that K is bounded between two positive numbers in $\mathbb{R}^n$, the conformal metric $g = u^{4/(n-2)}\, g_o$ is complete, and the volume growth of g can be arbitrarily fast or reasonably slow according to the constructions. By imposing natural conditions on u, we obtain growth estimate on the $L^{2n/(n-2)}$-norm of the solution and show that it has slow decay.

1. Introduction

In this article we derive L^p-estimates on positive solutions of the conformal scalar curvature equation

$$\Delta u + K u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $n \ge 3$ is an integer, Δ the standard Laplacian on $\mathbb{R}^n$, K a smooth function. Equation (1.1) relates the scalar curvature of the conformal metric $g = u^{4/(n-2)}\, g_o$ to $4K(n-1)/(n-2)$, where g_o is Euclidean metric [10]. It is assumed throughout this note that

$$0 < a^2 \le K(x) \le b^2 \quad \text{for large } |x| \tag{1.2}$$

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and for some positive constants a and b. The following estimates are known for any positive smooth solution u of equation (1.1) with condition (1.2).

$$\int_{S^{n-1}} u(r,\theta)\,d\theta \le C_1\, r^{\frac{2-n}{2}}\,, \tag{1.3}$$

$$\int_{B_o(r)} u^{\frac{n+2}{n-2}}(x)\,dx \le C_2\, r^{\frac{n-2}{2}} \tag{1.4}$$

for large r and for some positive constants C_1 and C_2 depending on u (see, for example, [11]). Here $B_o(r)$ is the ball with center at the origin and radius r, and S^{n-1} is the unit sphere in $\mathbb{R}^n$. We seek to obtain higher order estimates of the forms

$$\int_{S^{n-1}} u^p(r,\theta)\,d\theta \le C_3\, r^{(2-n)p/2}, \qquad p > 1\,; \tag{1.5}$$

$$\int_{B_o(r)} u^q(x)\,dx \le \begin{cases} C_4\, r^{n-(n-2)q/2} & \text{if} \quad q > \frac{n+2}{n-2}\,,\ q \ne \frac{2n}{n-2}\,; \\ C_5\, \ln r & \text{if} \quad q = \frac{2n}{n-2}\,, \end{cases} \tag{1.6}$$

for large r, where $C_3\,,\ C_4$ and C_5 are positive constants. The above estimates are based on the *slow decay* of u, that is,

$$u(x) \le C_6\, |x|^{(2-n)/2} \quad \text{for large} \quad |x|\,, \tag{1.7}$$

where C_6 is a positive constant.

Our first observation is that, in general, (1.5), (1.6) or (1.7) do not hold. Taliaferro [13] shows that positive solution of (1.1) outside a ball in $\mathbb{R}^n$ with condition (1.2) may not have slow decay. We modify the construction in [13] to obtain positive C^2-solutions of (1.1) in $\mathbb{R}^n$ with K bounded between two positive numbers in $\mathbb{R}^n$, such that the conformal metric $g = u^{4/(n-2)}\, g_o$ is complete and the volume growth of $(\mathbb{R}^n, g)$ can be arbitrarily fast or reasonably slow with respect to the constructions. This suggests that the geometric structure of complete manifolds of bounded positive scalar curvature could be very complicated (cf. [9]).

It is observed in [6] that if estimate (1.5) holds for some number $p \ge 2n/(n-2)$, then u has slow decay and hence (1.5) and (1.6) hold for all $p,\ q > 1$. The integral in estimate (1.6) is the volume growth of $(\mathbb{R}^n, g)$ when $q = 2n/(n-2)$. In order to obtain (1.5) and (1.6) for large p and q, additional conditions on K or u are required. By using a novel version of the moving plane method, Chen-Lin ([2] [3] and [4]) and Lin [12] examine, among other things, slow decay of u under the condition

$$0 \;<\; \frac{C_7}{|x|^{1+\alpha}} \;\le\; |\bigtriangledown K(x)| \;\le\; \frac{C_8}{|x|^{1+\alpha}} \quad \text{for large} \quad |x| \tag{1.8}$$

and for some positive constants α, C_7 and C_8.

To gain better understanding on u, consider the case when K is equal to a positive constant, say $K = n(n-2)/4$, outside a compact subset of $\mathbb{R}^n$. We express u as an associated function on the cylinder $\mathbb{R} \times S^{n-1}$ by letting

$$(1.9) \qquad v(s,\theta) = |x|^{\frac{n-2}{2}} u(x)\,, \quad \text{where } \ |x| = e^s \ \text{ and } \ \theta = x/|x| \in S^{n-1}.$$

Then v satisfies the equation

$$(1.10) \qquad \frac{\partial^2 v}{\partial s^2} + \Delta_\theta\, v - \frac{(n-2)^2}{4} v + K\, v^{\frac{n+2}{n-2}} = 0 \quad \text{in } \ \mathbb{R} \times S^{n-1},$$

where Δ_θ is the standard Laplacian on S^{n-1}. Here K is interpreted as a function on $\mathbb{R} \times S^{n-1}$ such that $(s,\theta) \mapsto K(e^s,\theta)$ for $s \in \mathbb{R}$ and $\theta \in S^{n-1}$. By a result of Caffarelli, Gidas and Spruck [1], with improvements by Korevaar, Mazzeo, Pacard and Schoen [8], either g can be realized as a smooth metric on S^n (in this case u is said to have *fast decay*), or

$$(1.11) \qquad v(s,\theta) = v_\varepsilon(s+T)\,[1 + O(e^{-\kappa s})] \quad \text{for} \quad \text{large } \ s\,, \ \ \theta \in S^{n-1}$$

and for some constants $\kappa > 0$ and $T \in \mathbb{R}$. Here v_ε, $\varepsilon \in (0,\ [(n-2)/n]^{(n-2)/4}]$, is one of a one-parameter family of positive solutions of the O.D.E.

$$(1.12) \qquad v'' - \frac{(n-2)^2}{4}\, v + \frac{n(n-2)}{4}\, v^{\frac{n+2}{n-2}} = 0 \quad \text{in } \ \mathbb{R}\,,$$

and $\varepsilon = \min\limits_{t \in \mathbb{R}} v(t)$ is referred as the necksize of the solution [8]. As O.D.E. (1.12) is autonomous, $|v_\varepsilon'|$ is bounded in $\mathbb{R}$. Furthermore, the Pohozaev number

$$(1.13) \ \ P(u) = \lim_{r \to +\infty} P(u,r) \quad \text{where} \quad P(u,r) = \frac{n-2}{2n} \int_{B_o(r)} x \cdot \nabla K(x)\, u^{\frac{2n}{n-2}}(x)\, dx$$

is a negative number [8]. When K may not be a constant outside a compact subset of $\mathbb{R}^n$, we have the following results.

Theorem A. *Let u be a positive smooth solution of equation (1.1) with condition (1.2), and v given by (1.9). Assume that there exist positive constants C_9 and C_{10} such that*

$$(1.14) \qquad \int_{S^{n-1}} \left(\frac{\partial v}{\partial s} \right)^2 (s,\theta)\, d\theta \,\le\, C_9 + C_{10} \int_{S^{n-1}} v^2(s,\theta)\, d\theta$$

for large s. If $P(u,r) \ge -\delta^2$ for large r and for a positive constant δ, then

$$(1.15) \qquad \int_{B_o(r)} u^{\frac{2n}{n-2}}\, dx \le C' \ln r \quad \textit{and} \quad \int_{B_o(r)} |\nabla u|^2\, dx \le C'' \ln r$$

for large r and for some positive constants C' and C''.

Theorem B. *Assumed that there exist positive constants C_{11} and C_{12} such that*

$$\left|\frac{\partial v}{\partial s}\right|(s,\theta) \le C_{11} + C_{12}\, v(s,\theta) \quad \textit{for large } s \textit{ and } \theta \in S^{n-1}. \tag{1.16}$$

If $P(u,r) \ge -\delta^2$ for large r and for a positive constant δ, then

$$\int_{S^{n-1}} u^{\frac{2n}{n-2}}(r,\theta)\, d\theta \le C\, r^{-n} \tag{1.17}$$

for large s and for some positive constant C. Moreover, u has slow decay.

We prove theorems A and B in section 4. Lower bounds on $P(u,r)$ are obtained in section 3, and examples are constructed in section 2. We use c, C, C_1, C_2 , ... to denote positive constants, which may be different from section to section.

2. Examples

We begin with a construction of positive C^2-solution u of equation (1.1) with K bounded between two positive constants in $\mathbb{R}^n$, such that u is unbounded from above in $\mathbb{R}^n$ (and hence does not have slow decay), and the conformal metric g is complete. Throughout this note $n \ge 3$ is an integer. Let

$$\bar u\,(r,\lambda) = \alpha_n \left(\frac{\lambda}{\lambda^2 + r^2}\right)^{(n-2)/2} \qquad \text{for } \ r \ge 0 \ \text{ and } \ \lambda > 0\,, \tag{2.1}$$

where $\alpha_n = [n(n-2)]^{(n-2)/4}$, and

$$u_o(x) = \bar u\,(|x|\,,1) = \frac{\alpha_n}{(1+|x|^2)^{(n-2)/2}} \qquad \text{for } \ x \in \mathbb{R}^n. \tag{2.2}$$

Let $\{\varepsilon_k\}_{k=1}^\infty \subset (0,1)$ be a sequence of decreasing numbers such that

$$\sum_{k=1}^\infty \varepsilon_k = 1\,, \tag{2.3}$$

$\{r_k\}_{k=1}^\infty$ a sequence of positive numbers such that $r_1 \ge 1$, $r_{k+1} - r_k \ge 1$ for $k = 1,\, 2, ...,$ and $\{M_k\}$ a sequence of positive numbers such that $M_k \to +\infty$ as $k \to +\infty$. For $x^{1,k} := (r_k,\, 0, ..., 0) \in \mathbb{R}^n$, $k = 1,\, 2, ...,$ there exist positive numbers λ_k , $k = 1,\, 2, ...,$ such that

$$u_k(x) := \bar u\,(|x - x^{1,k}|,\, \lambda_k) \qquad \text{for } \ x \in \mathbb{R}^n \tag{2.4}$$

satisfies

$$\Delta u_k + u_k^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n, \tag{2.5}$$

$$u_k(x) \le \varepsilon_k\, u_o(x) \quad \text{and} \quad |\bigtriangledown u_k(x)| < \varepsilon_k \quad \text{for} \quad |x - x^{1,k}| \ge \frac{1}{4}, \quad \text{and} \tag{2.6}$$

$$u_k(x^{1,k}) = \alpha_n\, \lambda_k^{(2-n)/2} \ge M_k \tag{2.7}$$

for $k = 1, 2, ...$ Using (2.3) and (2.6), it follows as in [13] that $\sum\limits_{k=0}^{\infty} u_k$ converges uniformly on compact subsets of $\mathbb{R}^n$ to a positive C^2-function. For a positive number b, let

$$\tilde{u}_b(x) = (|x|^2 + b^2)^{(2-n)/4} \quad \text{for } x \in \mathbb{R}^n. \tag{2.8}$$

We have

$$\Delta\, \tilde{u}_b + K_b\, \tilde{u}_b^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n, \tag{2.9}$$

where

$$K_b(x) = \frac{n(n-2)}{2} \left(1 - \frac{n+2}{2n} \frac{|x|^2}{|x|^2 + b^2} \right) \quad \text{for } x \in \mathbb{R}^n. \tag{2.10}$$

In particular

$$\frac{n(n-2)^2}{4n} \le K_b(x) \le \frac{n(n-2)}{2} \quad \text{for } x \in \mathbb{R}^n. \tag{2.11}$$

Let

$$u(x) = \tilde{u}_b(x) + \sum_{k=0}^{\infty} u_k(x) \quad \text{for } x \in \mathbb{R}^n. \tag{2.12}$$

It follows from (2.5), (2.9) and (2.11) that

$$-\Delta\, u(x) = \left[K_b(x) \tilde{u}_b^{\frac{n+2}{n-2}}(x) + \sum_{k=0}^{\infty} u_k^{\frac{n+2}{n-2}}(x) \right] \le \frac{n(n-2)}{2} u^{\frac{n+2}{n-2}}(x) \tag{2.13}$$

for $x \in \mathbb{R}^n$. Assume that $x \in B_{x_{k'}}(1/4)$ for some positive integer k'. Using (2.3), (2.6) and the inequality $(a+b)^p \le 2^{p-1}(a^p + b^p)$ for $a,\ b \ge 0$ and $p \ge 1$, we have

$$\begin{aligned} u^{\frac{n+2}{n-2}}(x) &= \left[\tilde{u}_b(x) + u_{k'}(x) + \sum_{k \ne k'} u_k(x) \right]^{\frac{n+2}{n-2}} \le [\tilde{u}_b(x) + u_{k'}(x) + 2u_o(x)]^{\frac{n+2}{n-2}} \\ &\le c_1 \left[\tilde{u}_b^{\frac{n+2}{n-2}}(x) + u_{k'}^{\frac{n+2}{n-2}}(x) + u_o^{\frac{n+2}{n-2}}(x) \right] \le -c_2\, \Delta u(x)\,, \end{aligned} \tag{2.14}$$

where c_1 and c_2 are positive constants depending on n only. Similar estimate holds for $x \notin B_{x_{k'}}(1/4)$ for $k' = 1, 2, ...,$ if we choose c_2 to be large enough, which depends on n only. Thus u satisfies the equation $\Delta u + K u^{(n+2)/(n-2)} = 0$ in $\mathbb{R}^n$, where

$$K(x) = [-\Delta u(x)]\, [u^{\frac{n+2}{n-2}}(x)]^{-1} \quad \text{for } \; x \in \mathbb{R}^n$$

is a continuous function which is bounded in $\mathbb{R}^n$ between two positive constants by (2.13) and (2.14). (2.7) shows that u is not bounded from above in $\mathbb{R}^n$. The conformal metric $u^{4/(n-2)} g_o$ is complete because

$$u^{4/(n-2)}(x) \ge \tilde{u}_b^{4/(n-2)}(x) \ge (1/2)|x|^{-2} \tag{2.15}$$

for large $|x|$. Let

$$V_n := \omega_n \int_0^\infty \left(\frac{\lambda \sqrt{n(n-2)}}{\lambda^2 + r^2} \right)^n r^{n-1}\, dr = \omega_n \int_0^\infty \left(\frac{\sqrt{n(n-2)}}{1 + t^2} \right)^n t^{n-1}\, dt \tag{2.16}$$

for $\lambda > 0$, where $t = \lambda^{-1} r$ and ω_n is the volume of the unit sphere in $\mathbb{R}^n$. By choosing r_k suitably far from each other, together with (2.16) and the fact that the first integral in (2.16) concentrates more on a neighborhood of 0 for smaller λ, we have

$$\int_{B_o(r)} u^{\frac{2n}{n-2}}(x)\, dx \le C_2 \ln r \tag{2.17}$$

for large r and for a positive constant C_2.

Next, given a function $\phi : [0, \infty) \to [0, \infty)$, we construct a positive C^2-solution u of equation (1.1) with K bounded between two positive constants in $\mathbb{R}^n$, such that the conformal metric $g = u^{4/(n-2)} g_o$ is complete and

$$\int_{B_o(r)} u^{\frac{2n}{n-2}}(x)\, dx \;\ge\; \phi(r) \quad \text{for } \; r > 2\,. \tag{2.18}$$

Without loss of generality, we may assume that ϕ is increasing and $\phi(0) \ge 10\, V_n$. For $k = 1, 2, ...,$ let N_k be a positive integer such that

$$N_k \ge 2\, V_n^{-1}\, \phi\,(k+2) \quad \text{for } \; k = 1, 2, ... \tag{2.19}$$

Let $\{\epsilon_k\}_{k=1}^\infty \subset (0, 1)$ be a sequence of decreasing numbers such that

$$\sum_{k=1}^\infty N_k\, \epsilon_k \;\le\; 1\,. \tag{2.20}$$

Let $\theta_k = 2\pi / N_k$. Let

$$x_{k,j} = (k \,\sin\,(j\,\theta_k)\,,\; k \cos\,(j\,\theta_k)\,,\; 0\,, ...,\; 0) \;\in \mathbb{R}^n \quad \text{for } \; j = 1, 2, ..., N_k\,, \tag{2.21}$$

and

$$(2.22) \qquad u_{k,j}(x) = \bar{u}\,(|x - x_{k,j}\,|,\,\lambda_k) \quad \text{for} \;\; x \in \mathbb{R}^n \quad \text{and} \;\; j = 1,\; 2, ..., N_k\,.$$

We choose λ_k to be small so that

$$(2.23) \;\; u_{k,j}(x) \le \epsilon_k\, u_o(x) \quad \text{and} \quad |\bigtriangledown u_{k,j}(x)\,| < \epsilon_k \quad \text{for} \quad |x - x_{k,j}\,| \ge \pi/(10N_k)\,,$$

and

$$(2.24) \qquad \int_{B_{x_{k,j}}(\pi/(10N_k))} u_{k,j}^{\frac{2n}{n-2}}(x)\,dx \ge \frac{V_n}{2} \quad \text{for} \;\; j = 1,\, 2, ..., N_k\,,$$

where $B_{x_{k,j}}(\pi/(10N_k))$ is the ball with center at $x_{k,j}$ and radius equal to $\pi/(10N_k)$. (2.24) is possible because, when λ is smaller, the first integral in (2.16) concentrates more on a neighborhood of the origin. It follows from (2.20) and (2.23) that the series $\sum\limits_{k=1}^{\infty}\sum\limits_{j=1}^{N_k} u_{k,j}$ converges uniformly on compact subsets of $\mathbb{R}^n$ to a positive C^2-function. Let

$$u = \tilde{u}_b + u_o + \sum_{k=1}^{\infty}\sum_{j=1}^{N_k} u_{k,j} \quad \text{in} \;\; \mathbb{R}^n.$$

As above, we have $\Delta u + K u^{(n+2)/(n-2)} = 0$ in $\mathbb{R}^n$, where K is a continuous function on $\mathbb{R}^n$ that is bounded between two positive constants. For any $r > 2$, let k be the integer such that $k+1 \le r < k+2$. By (2.19) we have

$$\begin{aligned} \phi(r) \;&\le\; \phi(k+2) \;\le\; \frac{V_n N_k}{2} \;\le\; \sum_{j=1}^{N_k} \int_{B_{x_{k,j}}(\pi/(10N_k))} u_{k,j}^{\frac{2n}{n-2}}(x)\,dx \\ &\le\; \int_{B_o(k+1)} u^{\frac{2n}{n-2}}(x)\,dx \;\le\; \int_{B_o(r)} u^{\frac{2n}{n-2}}(x)\,dx\,. \end{aligned}$$

3. Estimates on $P(u, r)$

Let $P(u, r)$ be given by (1.13) in the introduction. The Pohozaev identity (see, for example, [7]) states that

$$(3.1) \qquad P(u, r) = \int_{S_r} \left[r\left(\frac{\partial u}{\partial r}\right)^2 - \frac{r}{2}|\bigtriangledown u|^2 + \frac{n-2}{2n}\, rKu^{\frac{2n}{n-2}} + \frac{n-2}{2}\, u\,\frac{\partial u}{\partial r} \right] dS$$

for $r > 0$, where $S_r = \partial B_o(r)$ is the sphere of radius r.

Theorem 3.2. *Let u be a positive smooth solution of equation (1.1) with condition (1.2). Assume that u is bounded from above in $\mathbb{R}^n$ and*

$$\frac{\partial K}{\partial r}(x) \ \geq \ - \frac{C_1}{|x|^{(n+2)/2}\,(\ln\,|x|)^{1+\epsilon}} \tag{3.3}$$

for large $|x|$ and for some positive constants C_1 and ϵ. Then $P(u,r) \geq -\delta^2$ for large r and for a positive constant δ.

Proof. Fixing a large number R and using (1.4) we have

$$\begin{aligned}
& \int_{B_o((m+1)R)\setminus B_o(mR)} r\frac{\partial K}{\partial r}(x) u^{\frac{2n}{n-2}}(x)\,dx \\
\geq \ & -\frac{C_1}{(mR)^{\frac{n}{2}}[\ln\,(mR)]^{1+\epsilon}} \int_{B_o((m+1)R)\setminus B_o(mR)} u^{\frac{2n}{n-2}}(x)\,dx \\
\geq \ & -\frac{C_2}{(mR)^{\frac{n}{2}}[\ln\,(mR)]^{1+\epsilon}} \int_{B_o((m+1)R)\setminus B_o(mR)} u^{\frac{n+2}{n-2}}(x)\,dx \\
\geq \ & -\frac{C_3[(m+1)R]^{\frac{n-2}{2}}}{(mR)^{\frac{n}{2}}[\ln\,(mR)]^{1+\epsilon}} \ \geq \ -\frac{C_4}{m\,(\ln\, m)^{1+\epsilon}}
\end{aligned}$$

for any positive integer m larger than 1, where $r = |x|$. Here C_2, C_3 and C_4 are positive constants. As the series

$$\sum_{m=2}^{\infty} \frac{1}{m\,(\ln m)^{1+\epsilon}}$$

converges, we conclude that there exists a positive constant δ such that $P(u,r) \geq -\,\delta^2$ for large r. ☐

Theorem 3.4. *Let u be a positive smooth solution of equation (1.1) with condition (1.2). Assume that there exists a positive constant c such that*

$$\frac{\partial K}{\partial r}(r,\theta) \ \geq \ -\frac{c}{r^2} \quad for \ \ large \ \ r \ \ and \ \ \theta \in S^{n-1}. \tag{3.5}$$

If there exist positive constants C and $\lambda \in (0,1)$ such that

$$\int_{S^{n-1}} \left(\frac{\partial v}{\partial s}\right)^{\frac{2n}{n-2}} (s,\theta)\,d\theta \leq C\ e^{\lambda\, s} \tag{3.6}$$

for large s, then $P(u,r) \geq -\,\delta^2$ for large r and for a positive constant δ.

Proof. For a positive number $\varepsilon > 0$ such that $\varepsilon + \lambda < 1$, using Young's inequality we have

$$\begin{aligned}
& \frac{d}{dr}\left(\int_{S_r} r^{\varepsilon} u^{\frac{2n}{n-2}}(r,\theta)\, dS \right) \; = \; \frac{d}{dr}\left(\int_{S^{n-1}} r^{n-1+\varepsilon} u^{\frac{2n}{n-2}}(r,\theta)\, d\theta \right) \\
= \; & \frac{n-1+\varepsilon}{r} \int_{S^{n-1}} r^{n-1+\varepsilon} u^{\frac{2n}{n-2}}(r,\theta)\, d\theta + \frac{2n}{n-2} \int_{S^{n-1}} u^{\frac{n+2}{n-2}}(r,\theta) \frac{\partial u}{\partial r}(r,\theta)\, r^{n-1+\epsilon}\, d\theta \\
= \; & \frac{-1+\varepsilon}{r} \int_{S^{n-1}} r^{n-1+\varepsilon} u^{\frac{2n}{n-2}}(r,\theta)\, d\theta \\
& \quad + \frac{2n}{n-2} \int_{S^{n-1}} u^{\frac{n+2}{n-2}}(r,\theta) \left[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r} \right](r,\theta)\, r^{n-1+\epsilon}\, d\theta \\
\le \; & \frac{C_5}{r^{2-\varepsilon}} \int_{S^{n-1}} \left\{ r^{\frac{n}{2}} \left[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r} \right](r,\theta) \right\}^{\frac{2n}{n-2}} d\theta \\
= \; & \frac{C_5}{r^{2-\varepsilon}} \int_{S^{n-1}} \left(\frac{\partial v}{\partial s} \right)^{\frac{2n}{n-2}} (s,\theta)\, d\theta \; \le \; \frac{C_6}{r^{2-\lambda-\varepsilon}}
\end{aligned}$$

for large r, where $r = e^s$ and C_5 and C_6 are positive constants. It follows that there exists a positive constant C_7 such that

$$\int_{S_r} r^{\varepsilon} u^{\frac{2n}{n-2}}\, dS \le C_7 \quad \text{or} \quad \int_{S_r} u^{\frac{2n}{n-2}}\, dS \le C_7\, r^{-\varepsilon} \tag{3.7}$$

for large r. For a fixed large number R_o, we have

$$\frac{2n}{n-2}\, P(u,R) = \int_{B_o(R)} r \frac{\partial K}{\partial r} u^{\frac{2n}{n-2}} dx \ge -C_8 - C_9 \int_{R_o}^{R} r^{-1} \int_{S_r} u^{\frac{2n}{n-2}}\, dS\, dr \ge -C_{10}$$

for large R with $R_o < R$. Here C_8, C_9 and C_{10} are positive constants. □

4. Proofs of Theorem A and B

Proof of Theorem A. Let

$$w(s) = \frac{1}{2} \int_{S^{n-1}} v^2(s,\theta)\, d\theta \quad \text{for} \;\; s \in \mathbb{R}\,, \tag{4.1}$$

where v is defined in (1.9). Using equation (1.10) we have

$$\begin{aligned}
w''(s) \; = \; & \int_{S^{n-1}} \left(\frac{\partial v}{\partial s} \right)^2 (s,\theta)\, d\theta + \int_{S^{n-1}} |\, \nabla_\theta\, v\,(s,\theta)|^2\, d\theta \\
& + \left(\frac{n-2}{2} \right)^2 \int_{S^{n-1}} v^2(s,\theta)\, d\theta - \int_{S^{n-1}} K(e^s,\theta)\, v^{\frac{2n}{n-2}}(s,\theta)\, d\theta
\end{aligned} \tag{4.2}$$

for $s \in \mathbb{R}$. The Pohozaev identity can be expressed as

$$
\begin{aligned}
(4.3)\quad 2P(u, e^s) \;=\;& \int_{S^{n-1}} \left(\frac{\partial v}{\partial s}\right)^2 (s,\theta)\, d\theta - \int_{S^{n-1}} |\,\nabla_\theta\, v\,(s,\theta)|^2\, d\theta \\
& - \left(\frac{n-2}{2}\right)^2 \int_{S^{n-1}} v^2(s,\theta)\, d\theta + \frac{n-2}{n} \int_{S^{n-1}} K(e^s,\theta)\, v^{\frac{2n}{n-2}}(s,\theta)\, d\theta
\end{aligned}
$$

for $s \in \mathbb{R}$ [6]. It follows from (1.2), (1.4), (4.2) and (4.3) that

$$
(4.4)\qquad w''(s) \le C_1 + C_2 \int_{S^{n-1}} v^2(s,\theta)\, d\theta - \frac{2\,a^2}{n} \int_{S^{n-1}} v^{\frac{2n}{n-2}}(s,\theta)\, d\theta
$$

for large s, where C_1 and C_2 are positive constants. Applying Young's inequality we obtain

$$
(4.5)\qquad -\frac{2\,a^2}{n} \int_{S^{n-1}} v^{\frac{2n}{n-2}}(s,\theta)\, d\theta \le C_3 - C_4 \int_{S^{n-1}} v^2(s,\theta)\, d\theta
$$

for large s, where C_3 and C_4 are positive constants. Furthermore, by choosing C_3 to be large, we can take C_4 to be large as well. Hence there exists a positive constant C_5 such that

$$
(4.6)\qquad w''(s) \le C_5 - w(s) \quad \text{for} \quad \text{large} \;\; s\,.
$$

From (4.6) it is easy to see that $w(s)$ is uniformly bounded from above for large s. To prove this assertion, assume that there is a large s' such that $w(s') \ge C_5 + 1$. (4.6) implies that $w''(s') \le -1$. Let s_o be a number larger than s' such that $w(s_o) < C_5 + 1$ and $w'(s_o) \le 0$. If $w(s) < C_5 + 1$ for all $s > s_o$, then we are done. Assume that s_1 is the smallest number larger than s_o such that $w(s_1) = C_5 + 1$. We claim that

$$
(4.7)\qquad D := w'(s_1) < 2\,(C_5 + 1)\,.
$$

Let $\bar{s} \in (s_o, s_1)$ be the largest number such that $w'(\bar{s}) = D/2$. As $w'' \le C_5$ on $(s_o\,, s_1)$, we have $s_1 - \bar{s} \ge D/(2C_5)$. On the other hand, $w' \ge D/2$ on $(\bar{s}, s_1)$. Therefore we have

$$
C_5 + 1 \;\ge\; w(s_1) - w(\bar{s}) \;\ge\; \frac{D}{2C_5} \cdot \frac{D}{2} \;\;\Rightarrow\;\; D^2 \;\le\; 4C_5(C_5+1)\,.
$$

Hence we have (4.7). From s_1, $w(s)$ can become no larger than $(C_5{+}1){+}[2(C_5{+}1)]^2$ before $w'(s)$ becomes negative again. Hence we conclude that $w(s)$ is uniformly bounded from above for large s.

From Pohozaev identity (3.1) we obtain

(4.8)
$$\int_{S_r} r|\bigtriangledown u|^2\,dS = 2\int_{S_r}\left[r\left(\frac{\partial u}{\partial r}\right)^2 + \frac{n-2}{2n} rKu^{\frac{2n}{n-2}} + \frac{n-2}{2}\,u\,\frac{\partial u}{\partial r}\right]dS - 2P(u,r)$$

for $r>0$. We have

(4.9)
$$\int_{S_r} r\left(\frac{\partial u}{\partial r}\right)^2 dS = \int_{S_r} r\left[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r}\right]^2 dS - (n-2)\int_{S_r} u\frac{\partial u}{\partial r}\,dS - \left(\frac{n-2}{2}\right)^2\int_{S_r}\frac{u^2}{r}\,dS$$

for $r>0$. Using (1.14) and the fact that w is bounded from above we obtain

$$\begin{aligned}(4.10)\qquad -\int_{S_r} u\frac{\partial u}{\partial r}\,dS &= -\int_{S_r} u\left[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r}\right]dS + \frac{n-2}{2}\int_{S_r}\frac{u^2}{r}\,dS\\ &\le \int_{S_r} r\left[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r}\right]^2 dS + \frac{n}{2}\int_{S_r}\frac{u^2}{r}\,dS\\ &= \int_{S^{n-1}}\left(\frac{\partial v}{\partial s}\right)^2 (s,\theta)\,d\theta + \frac{n}{2}\int_{S^{n-1}} v^2(s,\theta)\,d\theta \le C_6\end{aligned}$$

for large r and for a positive constant C_6, where $r=e^s$. It follows from (4.8), (4.9) and (4.10) that

$$\int_{S_r} |\bigtriangledown u|^2\,dS \,\le\, \frac{C_7}{r} + \frac{n-2}{n}\int_{S_r} Ku^{\frac{2n}{n-2}}\,dS$$

for large r, where C_7 is a positive constant. Therefore we obtain

$$(4.11)\qquad \int_{B_o(r)} |\bigtriangledown u|^2\,dx \,\le\, C_8\ln r + \frac{n-2}{n}\int_{B_o(r)} Ku^{\frac{2n}{n-2}}\,dx$$

for large r and for a positive constant $C_8 \ge C_7$. On the other hand we have

$$\begin{aligned}\int_{B_o(r)} Ku^{\frac{2n}{n-2}}\,dx &= \int_{B_o(r)} u\,(-\Delta u)\,dx = \int_{B_o(r)} |\bigtriangledown u|^2\,dx - \int_{S_r} u\frac{\partial u}{\partial r}\,dS\\ &\le C_8\ln r + \frac{n-2}{n}\int_{B_o(r)} Ku^{\frac{2n}{n-2}}\,dx + C_6\end{aligned}$$

for large r, where we use (4.10). Hence there exists a positive constant C_9 such that

$$\int_{B_o(r)} Ku^{\frac{2n}{n-2}}\,dx \le C_9\,\ln r$$

for large r. If $u \in L^{2n/(n-2)}(\mathbb{R}^n)$, then clearly we have the first inequality in (1.15). Assume that $u \notin L^{2n/(n-2)}(\mathbb{R}^n)$. Using (1.2) we have

$$\int_{B_o(r)} u^{\frac{2n}{n-2}}\,dx \,\le\, \frac{2}{a^2}\int_{B_o(r)} Ku^{\frac{2n}{n-2}}\,dx \le \frac{2\,C_9}{a^2}\,\ln r$$

for large r. Hence we have the first inequality in (1.15). The second inequality follows from (4.11). $\square$

Proof of Theorem B. From the proof of theorem A we have

$$\int_{S_r} \frac{u^2(x)}{r}\, dS = \int_{S^{n-1}} v^2(s,\theta)\, d\theta = 2\, w(s) \le C_{10} \tag{4.12}$$

for large r, where $r = |x| = e^s$ and C_{10} is a positive constant. By using (1.16) and (4.12) we also have

(4.13)
$$\int_{S_r} r \left[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r} \right]^2 dS = \int_{S^{n-1}} r^n \left[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r} \right]^2 d\theta = \int_{S^{n-1}} \left(\frac{\partial v}{\partial s} \right)^2 d\theta \le C_{11}$$

for large r, where C_{11} is a positive constant. It follows from Pohozaev identity (3.1) that

$$\begin{aligned} \text{(4.14)}\ \int_{S_r} r|\bigtriangledown u|^2\, dS \ &\le\ C_{12} + \frac{n-2}{n}\int_{S_r} rKu^{\frac{2n}{n-2}}\, dS + 2\int_{S_r} r \left[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r} \right]^2 dS \\ &\qquad - (n-2)\int_{S_r} u \left[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r} \right] dS \\ &\le\ C_{13} + \frac{n-2}{n}\int_{S_r} rKu^{\frac{2n}{n-2}}\, dS + C_{14}\int_{S_r} r \left[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r} \right]^2 dS \\ &\qquad + C_{15}\int_{S_r} \frac{u^2}{r}\, dS \ \le\ C_{16} + \frac{n-2}{n}\int_{S_r} rKu^{\frac{2n}{n-2}}\, dS \end{aligned}$$

for large r, where we use (4.12) and (4.13). Here C_{12}, C_{13}, C_{14}, C_{15} and C_{16} are positive constants. (4.14) implies that there exists a positive constant C_{17} such that

$$\int_{B_o(R)} r|\bigtriangledown u|^2\, dx \ \le\ C_{17}\, R + \frac{n-2}{n}\int_{B_o(R)} rKu^{\frac{2n}{n-2}}\, dx \tag{4.15}$$

for large R. We have

$$\begin{aligned} \text{(4.16)}\qquad & \int_{B_o(R)} rKu^{\frac{2n}{n-2}}\, dx = \int_{B_o(R)} (ru)(-\Delta u)\, dx \\ &= \int_{B_o(R)} r|\bigtriangledown u|^2\, dx + \int_{B_o(R)} u\frac{\partial u}{\partial r}\, dx - R\int_{S_R} u\frac{\partial u}{\partial r}\, dS \end{aligned}$$

for $R > 0$. Using (4.13) we obtain

$$\begin{aligned} \text{(4.17)}\quad \int_{B_o(R)} u\,\frac{\partial u}{\partial r}\, dx \ &=\ \int_{B_o(R)} u \left[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r} \right] dx - \frac{n-2}{2}\int_{B_o(R)} \frac{u^2}{r}\, dx \\ &\le\ C_{18}\int_{B_o(R)} r \left[\frac{\partial u}{\partial r} + \frac{n-2}{2}\frac{u}{r} \right]^2 dx \ \le\ C_{19}\, R \end{aligned}$$

for large R, where C_{18} and C_{19} are positive constants. As in (4.10) we have

$$\left| R \int_{S_R} u \frac{\partial u}{\partial r}\, dS \right| \le C_{20}\, R \tag{4.18}$$

for large R, where C_{20} is a positive constant. From (4.15), (4.16), (4.17) and (4.18) we obtain

$$\frac{2}{n} \int_{B_o(R)} r K u^{\frac{2n}{n-2}}\, dx \le C_{21}\, R \tag{4.19}$$

for large R, where C_{21} is a positive constant. Using (1.16) we have

$$\begin{aligned}
& \frac{d}{dr}\left(\int_{S_r} r^2 u^{\frac{2n}{n-2}}\, dS \right) = \frac{d}{dr}\left(\int_{S^{n-1}} r^{n+1} u^{\frac{2n}{n-2}}\, d\theta \right) \\
= \ & (n+1) \int_{S^{n-1}} r^n u^{\frac{2n}{n-2}}\, d\theta + \frac{2n}{n-2} \int_{S^{n-1}} r^{n+1} u^{\frac{n+2}{n-2}} \frac{\partial u}{\partial r}\, d\theta \\
= \ & \int_{S^{n-1}} r^n u^{\frac{2n}{n-2}}\, d\theta + \frac{2n}{n-2} \int_{S^{n-1}} r^{n+1} u^{\frac{n+2}{n-2}} \left[\frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r} \right] d\theta \\
= \ & \int_{S^{n-1}} r^n u^{\frac{2n}{n-2}}\, d\theta + \frac{2n}{n-2} \int_{S^{n-1}} \left(r^{\frac{n+2}{2}} u^{\frac{n+2}{n-2}} \right) \left\{ r^{\frac{n}{2}} \left[\frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r} \right] \right\} d\theta \\
\le \ & C_{22} \int_{S^{n-1}} r^n u^{\frac{2n}{n-2}}\, d\theta + C_{23} \int_{S^{n-1}} \left\{ r^{\frac{n}{2}} \left[\frac{\partial u}{\partial r} + \frac{n-2}{2} \frac{u}{r} \right] \right\}^{\frac{2n}{n-2}} d\theta \\
= \ & C_{22} \int_{S^{n-1}} r^n u^{\frac{2n}{n-2}}\, d\theta + C_{23} \int_{S^{n-1}} \left| \frac{\partial v}{\partial s} \right|^{\frac{2n}{n-2}} d\theta \ \le \ C_{24} \int_{S^{n-1}} r^n u^{\frac{2n}{n-2}}\, d\theta + C_{25}
\end{aligned}$$

for large r, and for some positive constants C_{22}, C_{23}, C_{24} and C_{25}, where $r = e^s$. Hence

$$\begin{aligned}
\int_{S_R} R^2 u^{\frac{2n}{n-2}}\, dS \ &= \ \int_0^R \left(\int_{S_t} t^2 u^{\frac{2n}{n-2}}\, dS \right)' dt \ \le \ C_{26} + \int_{r_o}^R \left(\int_{S_t} t^2 u^{\frac{2n}{n-2}}\, dS \right)' dt \\
&\le \ C_{26} \, + \, C_{24} \int_{r_o}^R \int_{S_t} t u^{\frac{2n}{n-2}}\, dS\, dt + C_{25}(R - r_o) \\
&\le \ C_{27}\, R \, + \, C_{28} \int_{B_o(R)} r u^{\frac{2n}{n-2}}\, dx
\end{aligned} \tag{4.20}$$

for R and r_o large, with $R > r_o$. Here C_{26}, C_{27} and C_{28} are positive constants. Consider the case when $u \notin L^{2n/(n-2)}(\mathbb{R}^n)$. It follows from (1.2) and (4.19) that

$$\int_{B_o(R)} r u^{\frac{2n}{n-2}} dx \ \le \ C_{29} \int_{B_o(R)} r K u^{\frac{2n}{n-2}} dx \le C_{30}\, R \tag{4.21}$$

for large R and for some positive constants C_{29} and C_{30}. Clearly we have

$$\int_{B_o(R)} r u^{\frac{2n}{n-2}} dx \ \le \ C_{31}\, R \tag{4.22}$$

for large R and for some positive constants C_{31} if $u \in L^{2n/(n-2)}(\mathbb{R}^n)$. From (4.20), (4.21) and (4.22) we have (1.17). By the results in [11] (see also [6]), we obtain slow decay (1.7) as well. $\square$

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